Lecture Notes for January 14, 2010

What this course is about: Classic Arrow-Debreu general equilibrium model of the economy.

Economic General Equilibrium

General Equilibrium Theory: Who was Prof. Debreu and why did he have his own parking space in Berkeley's Central Campus??

Nobel Prizes: Arrow, Debreu

June 1993: A birthday party for mathematical general equilibrium theory!

October 2005: Mathematical Economics: The Legacy of Gerard Debreu
http://emlab.berkeley.edu/users/cshannon/debreu/home.htm

What does mathematical general equilibrium theory do? Tries to put microeconomics on same basis of logical precision as algebra or geometry. Axiomatic method: allows generalization; clearly distinguishes assumptions from conclusions and clarifies the links between them.

Two ideas about writing an economic theory:

Ockam's razor (KISS - Keep it simple, stupid.), improves generality

Precision, reliable results, Hugo Sonnenschein: "In 1954, referring to the first and second theorems of classical welfare economics, Gerard wrote 'The contents of both Theorems ... are old beliefs in economics. Arrow and Debreu have recently treated these questions with techniques permitting proofs.' This statement is precisely correct; once there were beliefs, now there was knowledge.

"But more was at stake. Great scholars change the way that we think about the world, and about what and who we are. The Arrow-Debreu model, as communicated in *Theory of Value* changed basic thinking, and it quickly became the standard model of price theory. It is the 'benchmark' model in Finance, International Trade, Public Finance, Transportation, and even macroeconomics. ... In rather short order it was no longer 'as it is' in Marshall, Hicks, and Samuelson; rather it became 'as it is' in *Theory of Value*." (remarks at the Debreu conference, Berkeley, 2005).

Partial and General Economic Equilibrium

PARTIAL EQUILIBRIUM

$S_k(p^o_k) = D_k(p^o_k)$, with $p^o_k > 0$ (or possibly, $p^o_k = 0$), or

$p^o_k = 0$ if $S_k(p^o_k) > D_k(p^o_k)$.

GENERAL EQUILIBRIUM For all i = 1,...,N,

$D_i(p^o_1, p^o_2,...,p^o_N) = S_i(p^o_1,..,p^o_N)$, $p^o_i > 0$, and

$p^o_i = 0$ for goods i such that

$$D_i(p^o_1,...,p^o_N) < S_i(p^o_1,...,p^o_N).$$

What's wrong with partial equilibrium? Suppose there's no consistent choice of $(p^o_1,..,p^o_N)$. Then there would be (apparent) partial equilibrium --- viewing each market separately --- but no way to sustain it, because of cross-market interaction. Competitive equilibrium is supposed to make efficient use of resources by minimizing costs and allowing optimizing consumer choice. But how do we know prices in other markets reflect underlying scarcity assuming "other things being equal" . If not, then apparently efficient equilibrium allocation may be wasteful. A valid notion of equilibrium and efficiency needs to take cross-market interaction into account.

Three big ideas

Equilibrium: S(p) = D(p)
Decentralization
Efficiency

The Robinson Crusoe Model

q = oyster production
c = oyster consumption
168 (hours per week) endowment
L = labor demanded
R = leisure demanded
168-R =labor supplied

$$q = F(L) \quad (2.1)$$

$$R = 168 - L \quad (2.2)$$

Centralized Allocation

We assume second order conditions so that local maxima are global maxima:

$$F'' < 0, \frac{\partial^2 u}{\partial c^2} < 0, \frac{\partial^2 u}{\partial R^2} < 0.$$

u(c,R) = u(F(L), 168 - L) (2.3)

$$\max_L \ u(F(L),\ 168 - L) \qquad (2.4)$$

$$\frac{d}{dL} u(F(L),\ 168 - L) = 0 \qquad (2.5)$$

$$u_c F' - u_R = 0 \qquad (2.6)$$

$$\left[-\frac{dq}{dR}\right]_{u=u\max} = \frac{u_R}{u_c} = F' \qquad (2.7)$$

Pareto efficient

$MRS_{R,c} = MRT_{R,q} (= RPT_{R,q})$

<u>Decentralized Allocation</u>

$$\Pi = F(L) - wL = q - wL \qquad (2.8)$$

Income:

$$Y = w \cdot 168 + \Pi \qquad (2.9)$$

Budget constraint:

Y = wR + c (2.10)

Equivalently, c = Y- wR = Π + wL = Π + w(168-R), a more conventional definition of a household budget constraint.

Firm profit maximization:

$$\Pi = q - wL \qquad (2.11)$$

$$\frac{d\Pi}{dL} = F' - w = 0, \text{so } F'(L^0) = w \qquad (2.14)$$

Household budget constraint:

$$wR + c = Y = \Pi^0 + w168 \qquad (2.15)$$

Choose c, R to maximize u(c, R) subject to (1.14). The Lagrangian is

V = u(c,R) - λ (Y - wR - c)

$$\frac{\partial V}{\partial c} = \frac{\partial u}{\partial c} + \lambda = 0$$

$$\frac{\partial V}{\partial R} = \frac{\partial u}{\partial R} + \lambda w = 0$$

Dividing through, we have

$$MRS_{R,c} = \left[-\frac{dc}{dR} \right]_{u=constant} = \frac{\frac{\partial u}{\partial R}}{\frac{\partial u}{\partial c}} = w \qquad (2.19)$$

$$wR + c = w168 + \Pi^0 \qquad (2.20)$$

$$c = w(168 - R) + \Pi^0 \qquad (2.21)$$

Walras' Law

Note that the Walras Law holds at all wage rates --- both in and out of equilibrium. It is not an equilibrium condition.

$$Y = w\cdot 168 + \Pi = w168 + q - wL \quad = wR + c \qquad (2.22)$$

$$0 = w(R - (168 - L)) + (c - q)$$

$$0 = w(R + L - 168) + (c - q) \qquad (2.23)$$

Definition : Market equilibrium. Market equilibrium consists of a wage rate w^0 so that at $w^0, q = c$ and $L = 168 - R$, where q, L are determined by firm profit maximizing decisions and c, R are determined by household utility maximization. (in a centralized solution L=168-R by definition; in a market allocation wages and prices should adjust so that as an equilibrium condition L will be equated to 168-R).

Profit maximization at w^0 implies $w^0 = F'(L^0)$.
Utility maximization at w^o implies

$$\frac{u_R(c^0, R^0)}{u_c(c^0, R^0)} = w^0 \qquad (2.24)$$

Market-clearing implies $R^0 = 168 - L^0, c^0 = F(L^0)$.

So combining (2.14) and (2.24), we have

$$F' = \frac{u_R}{u_c} \qquad (2.25)$$

which implies Pareto efficiency.

7. Functions

We describe a function f(•) as follows:
For each $x \in A$ there is $y \in B$ so that $y = f(x)$.
$f: A \to B$.
A = domain of f
B = range of f
graph of f = S $\subset A x B$, S = {(x, y) | y = f(x)}

Let $T \subset A$. $f(T) \equiv \{y \mid y = f(x), x \in T\}$ is the image of T under f.
$f^{-1}: B \to A$, f^{-1} is known as "f inverse"
$f^{-1}(y) = \{x \mid x \in A,\ y = f(x)\}$

7.1 Continuous Functions

Let $f: A \to B,\ A \subset R^m,\ B \subset R^p$.

The notion of continuity of a function is that there are no jumps in the function values. Small changes in the argument of the function (x) result in small changes in the value of the function (y=f(x)).

Let ε, $\delta(\varepsilon)$, be small positive real numbers; we use the functional notation $\delta(\varepsilon)$ to emphasize that the choice of δ depends on the value of ε. f is said to be **<u>continuous</u>** at $a \in A$ if

(i) for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $|x - a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon$, or equivalently,

(ii) $x^\nu \in A,\ \nu = 1, 2, \ldots,$ and $x^\nu \to a,$ implies $f(x^\nu) \to f(a)$.

Theorem 7.5: Let $f: A \to B$, f continuous. Let $S \subset B$, S closed. Then $f^{-1}(S)$ is closed.
Proof: Let $x^\nu \in f^{-1}(S)$, $x^\nu \to x^o$. We must show that $x^o \in f^{-1}(S)$. Continuity of f implies that $f(x^\nu) \to f(x^o)$. $f(x^\nu) \in S$, S closed, implies $f(x^o) \in S$. Thus $x^o \in f^{-1}(S)$. QED

Note that as a consequence of Thm 7.5, the inverse image under a continuous function of an open subset of the range is open (since the complement of a closed set is open).

Theorem 7.6: Let $f: A \to B$, f continuous. Let $S \subset A$, S compact. Then f(S) is compact.
Proof: We must show that f(S) is closed and bounded.
<u>Closed</u>: Let $y^\nu \in f(S)$, ν=1,2,..., $y^\nu \to y^o$. Show that $y^o \in f(S)$. There is $x^\nu \in S$, $x^\nu = f^{-1}(y^\nu)$. Take a convergent subsequence, relabel, and $x^\nu \to x^o \in S$ by closedness of S. But continuity of f implies that $f(x^\nu) \to f(x^o) = y^o \in f(S)$.
<u>Bounded</u>: For each $y \in f(S)$, let $C(y)=\{z \mid z \in B, |y-z|<\varepsilon\}$, an ε-ball about y. The family of sets $\{f^{-1}(C(y)) \mid y \in f(S)\}$ is an open cover of S (the inverse image of an open set under f is open since the inverse image of its complement --- a closed set --- is closed, Thm 2.6). There is a finite subcover. Hence f(S) is covered by a finite family of ε balls. f(S) is bounded. QED

Corollary 7.2: Let $f: A \to R$, f continuous, $S \subset A$, S compact, then there are $\bar{x}, \underline{x} \in S$ such that $f(\bar{x}) = \sup\{f(x) \mid x \in S\}$ and $f(\underline{x}) = \inf\{f(x) \mid x \in S\}$, where inf indicates greatest lower bound and sup indicates least upper bound.

Corollary 7.2 is very important for economic analysis. It provides sufficient conditions so that maximizing behavior takes on well defined values.

Homogeneous Functions

f: $R^p \rightarrow R^q$.
f is homogeneous of degree 0 if for every scalar (real number) $\lambda > 0$, we have $f(\lambda x) = f(x)$.
f is homogeneous of degree 1 if for every scalar $\lambda > 0$, we have $f(\lambda x) = \lambda f(x)$.